Handling Boundary Constraints for Numerical Optimization by Particle Swarm Flying in Periodic Search Space

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Abstract - The Periodic mode is analyzed together with two conventional boundary handling modes for particle swarm. By providing an infinite space that comprises periodic copies of original search space, it avoids possible disorganizing of particle swarm that is induced by the undesired mutations at the boundary. The results on benchmark functions show that particle swarm with Periodic mode is capable of improving the search performance significantly, by compared with that of conventional modes and other algorithms.

I. INTRODUCTION

The numerical optimization problems (NOP) can be defined as finding \( \tilde{x} \in S \subseteq \mathbb{R}^D \) such that

\[
\begin{align*}
    f(\tilde{x}) & = \min \{ f(\tilde{y}); \tilde{y} \in S \}, \\
    g_j(\tilde{x}) & \leq 0, \quad \text{for } j = 1, m
\end{align*}
\]

where \( \tilde{x} = (x_1, \ldots, x_D) \) ( \( 1 \leq d \leq D \), \( d \in \mathbb{Z} \) ), and \( x_i \in \mathbb{I} \), \( l_i \) and \( u_i \) are lower and upper values of the boundary constraints, respectively. \( f \) and \( g \) are functions on \( S \); \( S \) is a \( D \)-dimensional space defined as a Cartesian product of domains of variables \( x_i \)'s. The set of points, which satisfying all the constraint functions \( g \) are denoted as feasible space (\( S_f \)).

Particle swarm optimization (PSO) [6, 11] is a novel stochastic algorithm inspired by social behavior of swarms. Each agent, call particle, flies in a \( D \)-dimensional space \( S \) according to the historical experiences of its own and its colleagues. The location of the \( i \)-th ( \( 1 \leq i \leq N \) ) particle is represented as \( \tilde{x}_i = (x_{i1}, \ldots, x_{iD}) \). The best previous location of the \( i \)-th particle is recorded and represented as \( \tilde{p}_i = (p_{i1}, \ldots, p_{iD}) \), which is also called pbest. The index of the best pbest among all the particles is represented by the symbol \( g \). The location \( \tilde{p}_g \) is also called gbest. The velocity for the \( i \)-th particle is represented as \( \tilde{v}_i = (v_{i1}, \ldots, v_{iD}) \). At each time step, the \( j \)-th particle is manipulated according to the following equations for its \( d \)-th dimension [12]:

\[
\begin{align*}
    v_{jd} &= w \cdot v_{jd} + c_1 \cdot U_1(j) \cdot (p_{jg} - x_{jd}) + c_2 \cdot U_2(j) \cdot (gbest - x_{jd}) \\
    x_{jd} &= x_{jd} + v_{jd}
\end{align*}
\]

where \( w \) is inertia weight, \( c_1 \) and \( c_2 \) are acceleration constants, \( U_1(j) \) and \( U_2(j) \) are random real values between 0 and 1.

The particle swarm is trying to perform as a self-organizing system with extraordinarily flexibility [14]. A particle “surfs” it on waves [1, 13], which \( \tilde{y} \) and \( \tilde{x} \) are similar to kinetic and potential energy, respectively; and \( \tilde{p}_g \) serves as a quasi-gravity center for the swarm, especially at the end of the process.

To solving NOP, two issues should be handled: a) constraint functions and b) boundary constraints.

The constraint handling methods [2-5, 7-10] are employed for constructing the fitness landscape \( f(\tilde{x}) \) in the \( S \). To avoiding the laborious and often troublesome setting of the penalty coefficients in the methods based on penalty functions [9], and without requiring a starting point in \( S_f \) [5], the handling methods that following Deb’s criteria [2] is often applied [7, 15]: a) any feasible solution is preferred to any infeasible solution; b) among two feasible solutions, the one having better objective function value is preferred; c) among two infeasible solutions, the one having smaller constraint violation is preferred.

However, to satisfying boundary constraints, the conventional handling methods [7, 14], which are keeping the individuals lying inside the \( S \) seem not always suitable for the efficient flying of particle swarm.

The purpose of this paper is to study a robust boundary constraints handling method for particle swarm. In the next section, the drawbacks of conventional handling methods, which keep the particles flying inside the \( S \) are discussed. Then in section 3, the Periodic handling mode is realized by providing an infinite space for the flying of particle swarm, which is composed of periodic copies of original \( S \). In section 4, experimental results in different boundary handling methods and comparison with existing results by different algorithms [3, 4, 10] are reported and discussed. In the last section, we conclude the paper.

II. CONVENTIONAL BOUNDARY HANDLING

For the current swarm, it has a possible flying domain in next generation according to equations (2), as the space \( S_n \) shown in Fig. 1. Its center is closed to the location of the \( \tilde{p}_g \), which is a gravity center of swarm.

The gbest is always changing its location during the evolution, which leads the swarm to moving. It means that some particles may exceed the boundary, i.e. \( S_n \cap \tilde{S} \neq \Phi \).
(where $\Phi$ is null set), especially when the $gbest$ is closer to the boundary.

The conventional boundary handling methods are trying to keep the points inside the original $S$. This means for the particles that flying into the set $S_m \cap S$, such as $\bar{x}_i \not\in S$ in Fig. 1, is invalid and should be adjusted to the boundary. For the particles that flying into the set $S$, trying to keep the points inside the original $S$ by an additional mutation $M(\bar{x})$ appended on equations (2).

Figure 1. Schematic for conventional boundary handling modes.

Unlike other algorithms, such as genetic algorithms (GAs) [2, 3], evolution strategies [4], and differential evolution (DE) [12], etc., the status of the $i$th generation of each particle in swarm have direct effects on its status of the next generation. However, when the $gbest$ is close to the boundary, the undesired mutations $M(\bar{x})$ may become too frequently to keep the self-organizing of swarm dynamics, which is maintained by the nonlinear interactions in swarm by equations (2).

Conventionally, there have two mainly modes are employed: Boundary mode and Random mode [7].

A. Boundary mode

For the Boundary mode, the $d$th dimension of $\bar{x} \not\in S$ is mutated to the boundary (as $\bar{x}_d \to \bar{y}_d$ in Fig. 1):

$$
\tilde{M}_d(x_d): \begin{cases} 
   x_d = l_d & \text{IF } x_d < l_d \\
   x_d = u_d & \text{IF } x_d > u_d
\end{cases}
$$

The $\tilde{M}_d(x_d)$ forces the particle that expects to outside the $S$ returns to the boundary, its direct effects of include: a) decreasing the $v_{ud}$; b) decreasing $p_{sr}\cdot x_{id}$ and $p_{sr}\cdot x_{id}$. Both decrease the "energy" of particles, i.e. decrease the domain of $S_m$. Moreover, for worse case, the point mutated by $\tilde{M}_d$ may become the $gbest$, which may be especially occurred at the early evolution stage, since the following reasons: a) the particles would have enough "energy" values to exceed the original $S$; b) the current $gbest$ is not in high fitness value. Then as a quasi-gravity center, it attracts other particles to the boundary and such particles cannot leave the boundary in the following generations unless the $gbest$ leave the boundary, which accelerates the swarm into equilibrium state and may lead to the premature convergence.

B. Random mode

For the Random mode, the $d$th dimension of $\bar{x} \not\in S$ is mutated with random values (as $\bar{x}_d \to \bar{y}_d$ in Fig. 1) [7]:

$$
\tilde{M}_d(x_d): \bar{x}_d = U_h(l_d, u_d) \quad \text{IF } x_d \not\in [l_d, u_d]
$$

where $U_h(l_d, u_d)$ is a random value between $l_d$ and $u_d$.

The point that is mutated by $\tilde{M}_d$ has the probability $(S_m \cap S)/S$ to exceed the $S_m$ since it is a random point in $S$. As the point exceeds the $S_m$, the direct effects of $\tilde{M}_d(x_d)$ include: a) increasing the $v_{id}$; b) increasing $p_{sr}\cdot x_{id}$ and $p_{sr}\cdot x_{id}$. Both effects increase the "energy" of swarm. If the global optimum is closed to the boundary, then $gbest$ is closing to the boundary on if the algorithm has capability to approach the global optimum. However, such frequent $M_d$ operations keep a relative large $S_m$, which disturbs the swarm into chaos state and decrease the convergence speed to global optimum.

III. PERIODIC SEARCH SPACE

This paper studies a new boundary handling method, which call as Periodic mode [15]. It provides an infinite search space for the flying of particles, which is composed of the periodic copies of original $S$ with same fitness landscape, as shown in Fig. 2. Where the grey region represents the original space $S^{(0)}=S$, its neighborhood regions are its periodic copies $S^{(i)}$.

Figure 2. Schematic for Periodic mode.

For the Periodic mode, the location $\bar{x}$ of each particle is not adjusted to $S$ by any mutation operations while...
\( \bar{x} \notin S \). However, \( F(\bar{x}) = F(\bar{z}) \), which \( \bar{z} \in S \) is the mapping point of \( \bar{x} \):

\[
\begin{align*}
M(x, z) &= \begin{cases} 
    z = x - (I_s - x_p) g_{ss} & \text{IF } x_s < I_s \\
    z = I_s + (x - u_l) g_s & \text{IF } x_s > u_l \\
    z = x_s & \text{IF } x_s \in [u_l, u_u] 
\end{cases} \\
\text{where } \% & \text{ is the modulus operator, } s_d = |u_d - I_d| \text{ is the parameter range of the } d\text{th dimension. The ultimate}\n\text{optimized point } \bar{x}^* \text{ is calculated from } M(x, \bar{x}^*) \text{, which is satisfying the boundary}\n\text{constraints.}
\end{align*}
\]

In Periodic mode, for \( \forall \bar{x} \), there exists an effective copy (which is called \( S^E \) as in Fig. 2) of \( S \), which the center point is \( \bar{x} \), and for the \( d\)th dimension, the lower and upper boundary values are \( x_{sd} = x_d / 2 \) and \( x_{sd} = x_d / 2 \), respectively. For each point \( \bar{x} \) in \( S^E \), it has a one-to-one mapping point that locating in the original \( S \). To searching in \( S^E \) is almost equivalent to searching in \( S \).

Comparing with conventional boundary constraints handling methods, the Periodic mode have some advantages to enhance robustness of the particle swarm.

Firstly, to eliminate the undesired mutations caused by boundary constraints, the ratio \( R(S_n, S) = (S_n \cap \overline{S}) / S_n \) that outside the center of \( S_n \) (often near the gbest) is located near or at the center of \( S \). It is possible for Periodic mode, since the searching space can be \( S^E \) of the gbest.

Secondly, the performance of an algorithm is improved if the variation distance between the current gbest and the global optimum point is shorter, especially when the gbest is temporarily trapped into a local optimum at a certain stage of evolution. In fact, for \( d\)th dimension, the maximum possible length is decreased from \( s_d \) (in the original \( S \) for conventional handling methods) to \( s_d / 2 \) (in the \( S^E \) of gbest for Periodic mode), as shown in Fig. 2.

Of course, it should be avoid that the domain of \( S_n \) being much larger than of \( S^E \) of the gbest in some dimensions, i.e. \( s_{n_d} = k_s s_d \) and \( k \) is much larger than 1. Since at the situation, although the particle swarm still maintains the self-organization dynamics, it has less efficiency due to the many redundant states in \( s_{n_d} \). Fortunately, it can be avoid as the particle swarm is convergent [1, 13], by using the following parameter settings: a) constriction factor [1]; and b) a small \( \nu \) during [14] or at least after the early stage [11] of evolution.

IV. RESULTS AND DISCUSSION

In order to study the performance of the boundary handling methods for particle swarm, eight constrained numerical problems by Michalewicz et al. [8] and four engineering design examples [10] have been tested, which were compared with recently published results[3, 4, 10].

A. Michalewicz’s examples [8]

Table 1 gives the global optimum (type and value \( F^* \)) and the published results by EAs [3, 4] for Michalewicz’s examples with inequality constraints [8]. Here for \( G_1, G_6, G_9 \), their global optima are located at the boundary; for \( G_6, G_7, G_9 \), their global optima are closed to the boundary.

| TABLE 1. GLOBAL OPTIMUM AND EXISTING RESULTS |
|---|---|---|---|---|
| \( F \) | Type | \( F^* \) | ES [4] | GA [3] |
| \( G_1 \) | Min | -15.000 | -15.000 |
| \( G_2 \) | Max | 0.80362 | 0.56 | 0.7901 |
| \( G_3 \) | Min | -30665.5 | -30665.5 | -30665.2 |
| \( G_4 \) | Min | -6961.814 | -6961.81 | -6961.8 |
| \( G_5 \) | Min | 24.306 | 24.6162 | 26.580 |
| \( G_6 \) | Max | 0.095825 | 0.095825 | 0.095825 |
| \( G_7 \) | Min | 680.705 | 680.705 | 680.72 |
| \( G_{10} \) | Min | 7049.248 | 7193.11 | 7267.89 |

The main parameters of \( (\mu + \lambda) \)-ES [4] includes: \( \mu = 100 \), \( \lambda = 300 \), the maximum number of generations \( T \lt 5 \cdot E 3 \), its total evaluation times \( T_E \approx 1.5 E 6 \).

The main parameters of GA [3] includes: the population size \( N = 70 \), \( T = 2E4 \), \( T_E = N \cdot T = 1.4E6 \).

Two particle swarm versions were tested: a) LPS [11]: with a linear decreasing inertia weight; b) DEPS [15]: hybrid with a differential evolution (DE) operator [12].

The parameters of LPS includes: a linearly decreasing \( \nu \) from 0.9 to 0.4 [11], \( c_1 = c_2 = 2 \), for \( d \)th dimension, the maximum velocity \( v_{max} = (u_d - I_d) / 2 \).

The parameters of DEPS includes: number of particles \( N \), \( c_1 = c_2 = 2.05 \) in constriction factor [1]; and for the hybrid DE operator [15], \( CR = 0.9 \).

For both particle swarm versions, four cases were tested, as in table 2, where \( T = 2E3 \). For all the cases, \( T_E = N \cdot T \) are less than the cases of GA [3] and ES [4]. For each example, 100 runs were executed.

| TABLE 2. SETTINGS FOR TEST CASES OF LPS AND DEPS |
|---|---|---|---|---|
| Cases | #B | #R | #P1 | #P2 |
| Mode | Boundary | Random | Periodic | Periodic |
| \( N \) | 14 | 14 | 14 | 70 |
| \( T_E \) | 28000 | 28000 | 28000 | 140000 |

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| TABLE 3. SUMMARY OF MEAN RESULTS BY THE CASES OF LPS |
|---|---|---|---|---|---|
| \( F \) | LPS/B (\( r_f \)) | LPS/R (\( r_f \)) | LPS/P1 | LPS/P2 |
| \( G_1 \) | -4.998 | -1.936 (79) | -14.9140 | -14.9981 |
| \( G_2 \) | 0.50052 | 0.43463 | 0.71921 | 0.77867 |
| \( G_3 \) | -30549.87 | -30517.0 | -30665.5 | -30665.4 |
| \( G_4 \) | -6961.8 (96) | -4592.9 | -6961.7 | -6961.814 |
| \( G_5 \) | 914.790 (11) | 38.511 | 26.047 | 25.161 |
| \( G_6 \) | 0.095825 (2) | 0.095825 | 0.095825 | 0.095825 |
| \( G_7 \) | 12114.09 | 680.76 | 680.75 | 680.66 |
| \( G_{10} \) | 12398 (21) | 8285.6 | 7756.6 | 7562.6 |

Table 3 gives the summary of mean best results \( F_B \) by the cases of LPS. The values \( r_f \) in the parenthesis gives the
ratio of runs that are failed in entering $S_r$ (if there have no parenthesis, it means no failed runs) in percentage, and only those runs that are succeeded in entering $S_r$ are counted for calculating the $F_p$.

For LPS/B, LPS/R, and LPS/P1, the only difference is their boundary handling modes. It can be found that LPS/B got worse results (and a large amount of runs are even failed in entering $S_r$, especially for $G_6$, 96% runs are failed), since many runs for Boundary mode were premature convergence at the boundary of $S$ during the early evolution stage. For LPS/R, it could not get satisfied results in give $T$ generations due to the unnecessarily $M_E$ operations in Random mode, when the global optimum of a problem is located at or closed to the boundary (except for $G_6$, $G_8$, $G_{10}$), especially for $G_6$, 79% runs are failed in entering $S_r$. For LPS/P1, it performs better than LPS/B and LPS/R in almost all examples.

LPS/P2 gets better results than LPS/P1 when $N$ is increased from 14 to 70. By comparison it with the existing results by EAs in Table 2, it can be seen that LPS/P2 get worse results than ES [4], which with one better ($G_5$), three almost same ($G_4$, $G_5$, $G_3$) and four worse ($G_1$, $G_7$, $G_6$, $G_{10}$) examples, while get better results than GA [3], which with four better ($G_4$, $G_5$, $G_8$, $G_{10}$), two almost same ($G_4$, $G_5$) and two worse ($G_6$, $G_{10}$) examples.

Table 4 gives the summary of mean best results $F_p$ by the cases of DEPS. The Boundary mode and Random mode of DEPS are better than that of LPS in most examples through the complementary searching role by the hybrid DE operator. Moreover, the DEPS/P1 still performs well than the DEPS/B and the DEPS/R. By comparison the DEPS/P2 with the existing results in the Table 2, it can be seen that DEPS/P2 get better results than ES [4] and GA [3] in almost all examples.

<table>
<thead>
<tr>
<th>$F$</th>
<th>DEPS/B ($\tau_c$)</th>
<th>DEPS/R</th>
<th>DEPS/P1</th>
<th>DEPS/P2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>-6.259</td>
<td>-12.248</td>
<td>-14.271</td>
<td>-15.000</td>
</tr>
<tr>
<td>$G_2$</td>
<td>0.36326</td>
<td>0.40280</td>
<td>0.48664</td>
<td>0.64330</td>
</tr>
<tr>
<td>$G_3$</td>
<td>-30646.43</td>
<td>-30662.20</td>
<td>-30665.54</td>
<td>-30665.54</td>
</tr>
<tr>
<td>$G_4$</td>
<td>-6961.8</td>
<td>-6961.814</td>
<td>-6961.814</td>
<td>-6961.814</td>
</tr>
<tr>
<td>$G_5$</td>
<td>209.300</td>
<td>263.89646</td>
<td>263.89646</td>
<td>263.89646</td>
</tr>
<tr>
<td>$G_6$</td>
<td>0.095691</td>
<td>0.095425</td>
<td>0.095558</td>
<td>0.095825</td>
</tr>
<tr>
<td>$G_7$</td>
<td>20819.319</td>
<td>680.690</td>
<td>680.690</td>
<td>680.630</td>
</tr>
<tr>
<td>$G_{10}$</td>
<td>8378.4</td>
<td>7506.5</td>
<td>7343.5</td>
<td>7049.5</td>
</tr>
</tbody>
</table>

Here the GA [3] performs the best for $G_2$ in all cases of algorithm settings. However, if the $T$ of LPS/P2 is increased to $1E4$, i.e. $T_E$ is increased to $7E5$, then $F_2$ of $G_2$ was 0.79298, which is better than GA [3].

### B. Engineering design examples [10]

Table 5 gives the global minimum value $F^*$ and existing results with number of evaluations by Ray et al. [10] for engineering design examples. Here the global optimum of $S_r$ is located at the boundary; for $WB$, $TS$, their global optimums are closed to the boundary.

Table 6 lists the results calculated by LPS in different handling modes, where $N=40$, $T_E=500$, then $T_E=2E4$ for each example, 500 runs were executed. Here the solutions of Boundary mode also are often trapped into the local minimums at the boundary. For Random mode, it found worse results for the cases $WB$, $TS$ and $SR$, which the global minimum are located at or closed to the boundary. Moreover, the Periodic mode can find better results that Ray’s [10] in less evaluation times.

<table>
<thead>
<tr>
<th>Design problems</th>
<th>$F^*$</th>
<th>Results [10]</th>
<th>$T_E$ [10]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Welded Beam (WB)</td>
<td>2.38113</td>
<td>2.96070</td>
<td>66862</td>
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<tr>
<td>Speed Reducer (SR)</td>
<td>2994.471</td>
<td>2998.027</td>
<td>110235</td>
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<tr>
<td>Three-Bar Truss (TB)</td>
<td>263.8958</td>
<td>263.8989</td>
<td>36113</td>
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<tr>
<td>Tension Spring (TS)</td>
<td>0.012666</td>
<td>0.012923</td>
<td>25167</td>
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</table>

<table>
<thead>
<tr>
<th>Mode</th>
<th>WB</th>
<th>SR</th>
<th>TB</th>
<th>TS</th>
</tr>
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<tbody>
<tr>
<td>Boundary</td>
<td>3.35408</td>
<td>3060.910</td>
<td>263.89646</td>
<td>0.013129</td>
</tr>
<tr>
<td>Random</td>
<td>2.56145</td>
<td>3106.733</td>
<td>263.89649</td>
<td>0.015372</td>
</tr>
<tr>
<td>Periodic</td>
<td>2.40403</td>
<td>2994.497</td>
<td>263.89654</td>
<td>0.012922</td>
</tr>
</tbody>
</table>

### V. Conclusion

This paper has analyzed a Periodic boundary handling mode, which is employed for improving the robustness of particle swarm. The method does not introduce any additional parameters. By providing an infinite space, which is composed of periodic copies of original $S$, it eliminates possible disorganizing for the particle swarm that caused by the unnecessary mutations at the boundary of $S$ as in conventional handling methods. Besides, it provides an effective copy of $S$ for the flying of dynamic particle swarm, which the maximum possible variation length by Periodic mode is also decreased to half of conventional handling methods for each dimension.

The performance of particle swarm with Periodic mode on benchmark functions was compared with that of conventional handling modes, include Boundary and Random mode, which produced better results, especially for the cases with local optimums that closed to or located at the boundary of $S$. It was also compared with the existing results of different algorithms, which provided better results in less evaluation times.

### REFERENCES


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